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**The Prices of Anarchy, Information and Cooperation.
An Application of Dynamic Game Theory to the Law**

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Abstract

The idea of quantifying the gap between the social optimum and game equilibria that is key for the design of efficient laws led to several research results in the recent past. The price of anarchy quantifies the loss of efficiency due to non-cooperation. It was introduced in 2004 and has been applied to both static and dynamic games since then. The price of information captures the different outcomes of games under different information structures and the price of cooperation measures the benefit or loss of a player due to altruistic behavior. Both indices were proposed for differential games in 2011.

In this thesis, we argue that the price of cooperation is hardly tractable even for very simple differential games. As an alternative we introduce the simple price of cooperation that measures the benefit or loss of a player due to altruistic behavior when he cannot observe the actions taken by the other players. For this index we are able to deduced several (tight) bounds for a variety of scalar linear quadratic differential games with 2 and N players, respectively.

Additionally we discuss errors occurring in some proofs of a recent publication on the prices of anarchy, information and cooperation and point out the consequences of these errors for some of the main results of the correspondent paper.

In summary we improve the methods to measure the effects of cooperation and information in a dynamic setting. As cooperation, information and altruistic behavior are key concepts for the law, we think that our thesis provides a valuable contribution to the theory of law.

Zusammenfassung

Die Idee der Quantifizierung der Lücke zwischen sozialem Optimum und Gleichgewichtslösungen von Spielen ist von besonderer Relevanz für den Entwurf von effizienten Gesetzen und führte zu etlichen Forschungsergebnissen in der jüngeren Vergangenheit. Der Preis der Anarchie quantifiziert den Verlust von Effizienz auf Grund von fehlender Kooperation. Er wurde 2004 in der Literatur eingeführt und seitdem auf statische und dynamische Spiele angewandt. Der Preis der Information vergleicht die verschiedenen Ergebnisse von Spielen unter verschiedenen Informationsstrukturen und der Preis der Kooperation misst den Gewinn oder Verlust eines Spielers auf Grund von altruistischem Verhalten. Beide Kennzahlen wurden 2011 im Kontext von Differentialspielen vorgeschlagen.

In dieser Arbeit argumentieren wir, dass der Preis der Kooperation sogar für sehr einfache Differentialspiele kaum explizit berechenbar ist. Als eine Alternative führen wir den einfachen Preis der Kooperation ein, welcher den Gewinn oder Verlust eines Spielers auf Grund von altruistischem Verhalten misst, wenn man zusätzlich annimmt, dass dieser Spieler die von den anderen Spielern vorgenommenen Handlungen nicht beobachten kann. Für diese Kennzahl gelingt es uns eine Reihe von (scharfen) Schranken für eine Vielzahl von skalaren linear-quadratischen Differentialspielen mit 2 und N Spielern abzuleiten.

Zusätzlich zeigen wir Fehler in einigen Beweisen aus einer kürzlich veröffentlichten Forschungsarbeit über die Preise der Anarchie, der Information und der Kooperation auf. Wir diskutieren dabei auch die aus diesen Fehlern resultierenden Konsequenzen für einige der Hauptresultate des entsprechenden Papers.

Zusammenfassend verbessern wir die Methoden um Effekte von Kooperation und Information in dynamischen Situationen zu messen. Da Kooperation, Information und altruistisches Verhalten zentrale Konzepte im Bereich der Rechtswissenschaften sind, glauben wir, dass die vorliegende Arbeit einen wertvollen Beitrag für diesen Bereich liefert.

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Chapter 1

Introduction

Game theory is a study of strategic behavior and strategic decision making. Strategic behavior arises whenever two or more individuals interact and each individual's decision influences the decisions (and the current situation and outcome) of all the individuals involved. More formally put, game theory is “the study of mathematical models of conflict and cooperation between intelligent rational decision-makers” [19]. Modern game theory began with John von Neumann's proof of the existence of mixed-strategy equilibria in two-person zero-sum games where he used Brouwer's fixed-point theorem on continuous mappings into compact convex sets, which became a standard method in game theory and mathematical economics [25, 26]. In 1950 John Nash proposed (in his dissertation) a criterion for mutual consistency of players' strategies, known as Nash equilibrium, applicable to a wider variety of games (N -person non-cooperative nonzero-sum games) than the criterion proposed by von Neumann [20]. In the 1960s, Reinhard Selten further refined the Nash equilibrium concept by introducing subgame perfect equilibria [23] and John Harsanyi developed the concepts of complete information and Bayesian games [15]. Nash, Selten and Harsanyi became Economics Nobel Laureates in 1994 for their contributions to economic game theory. Today game theory has become a very large, diverse and active field of research. It is mainly used in economics, political science, and psychology, but has also many applications in biology and law.

Laws affect the way people behave. Hence laws are often of importance in situations in which strategic behavior occurs in order to influence the individuals involved in an intended way (e.g. such that the outcome of the situation is a desired one in terms of common welfare). There exist several articles and books stressing the importance of game theory for the law as formal tool that can offer important insights in the way laws should and should not be formulated [2, 3, 7, 18, 21]. These books and articles also point out various fields of application, among them tort theory, contract law, antitrust law, bankruptcy law, employment law, and labor law.

It is not the aim of this thesis to give an introduction to or overview of game theory and law. For this propose we refer to [7, 10, 13, 17, 21, 26] and further references therein. What we aim for is to present corrections and extensions to one particular current research topic that belongs to the field “Game Theory and Law”. We will take the current research paper “Prices of Anarchy, Information, and Cooperation in Differential Games” by Bařar and Zhu [8], point out some errors and several typos therein and then extend their work by introducing the “simple price of cooperation (sPoC)” and deducing a variety of bounds on the sPoC for several differential games. For an introduction to and good overview of the theory of differential (and difference) games we refer to [6, 9, 12].

The price of anarchy (PoA) quantifies the loss of efficiency due to non-cooperation. It has been widely used in static games [16, 22] and was applied to dynamic games in [8] for the first time. Additionally Bařar and Zhu [8] introduced the price of information (PoI) that captures the different outcomes of games under different information structures and the price of cooperation (PoC) that measures the benefit or loss of a player due to altruistic behavior. The sPoC is a variant of the PoC where the altruistic players cannot

observe the actions taken by the other players. Hence the cooperating players only take into account the costs of the other players that result from their state variables (and ignore the costs resulting from the control variables of the other players). Contrary to the PoC that is hardly tractable even for very simple differential games we can deduce several (tight) bounds of the sPoC for a variety of scalar linear quadratic differential games with 2 and N players, respectively. Cooperation, information and altruistic behavior are key concepts for the law and hence improving the methods to measure these quantities in terms of the PoA, the PoI and the sPoC in a dynamic setting is a worthwhile contribution to the theory of law.

This thesis will be handed in to the department of “Law and Economics”. Recently Richard H. McAdams complained in his paper [18] that “legal scholars are nearly obsessed with the Prisoners’ Dilemma, mentioning the game in a staggering number of law review articles (over three thousand), while virtually ignoring other equally simple games offering equally sharp insights into legal problems. One has to guess that the former obsession contributes to the latter neglect.” In this spirit we hope that our thesis will help to extend the tools considered for legal analysis to simple differential games because we believe that in the long run game theory could transform legal theory as it has transformed economic theory.

The structure of the thesis is as follows. In Section 2 we provide some additional motivation for dealing with the prices of anarchy, information, and cooperation and recall the definitions and notation used in [8]. In Section 3 we discuss errors occurring in some proofs in [8] and point out the consequences of these errors for the main results of the corresponding paper. In the end of that section we additionally state a list of typos in [8]. Finally we introduce the sPoC in Section 4. In this main section of the thesis we deduce several (tight) bounds for the sPoC for a variety of scalar linear quadratic differential games with 2 and N players, respectively.

Chapter 2

Motivation and General Problem Formulation

It is well known that the non-cooperative Nash equilibrium in nonzero-sum games (even if it is unique) is inefficient in general [11]. Thus all players could possibly lower their costs simultaneously through a cooperative behavior. Often rules and laws are designed to invoke such a cooperative behavior. In dynamic and differential games that are a special class of non-cooperative nonzero-sum games information is (like in the area of law) a crucial issue. In this kind of games additional information can help but also (and that's counter-intuitive at the outset) hurt (for details see [4, 5, 6]). Hence one question of interest is to measure the extent of inefficiency or the effect of information in differential games (e.g. to decide whether it is pays off to install particular legal rules). The PoA was introduced in [22] for quantifying the loss of efficiency due to competition in traffic routing games. It has been shown that the PoA can be bounded by a constant for certain types of static games. Hence the players achieve at least some level of efficiency despite being suboptimal (for details see [22] and [16]).

The idea of quantifying the gap between the social optimum and game equilibria led to several research results in the same vein. The price of simplicity [24], the price of uncertainty [14] and the price of leadership [27] were proposed for different types of static games in the field of communication networks.

In [8] the PoA was generalized to differential games (for an exact definition see below). Furthermore they introduced the PoI that compares the equilibrium costs under different information structures and the PoC that measures benefit or loss of a player compared to his base Nash equilibrium payoff due to cooperation (again exact definitions are given below).

In the following we introduce the general nonzero-sum differential games framework (that is well developed, see e.g. [6, 9, 12]) along with the Nash equilibrium solution and the prices of anarchy, information, and cooperation. We use exactly the same notation as in [8] in order to facilitate the review of some results obtained in [8] in Section 3.

Let $\mathcal{N} = \{1, 2, \dots, N\}$ be the set of players, and $[0, T)$ be the time interval of interest. At each time instant $t \in [t, T)$, each player, say Player i , chooses an m_i -dimensional control value (action) $u_i(t)$ from his set of feasible control values $U_i \subset \mathbb{R}^{m_i}$. The state variable x is of dimension N , and takes values in \mathbb{R}^n . We also make the standard assumption that $u_i(\cdot), i \in \mathcal{N}$ is piecewise continuous and $x(\cdot)$ is piecewise continuously differentiable on $t \in [t, T)$. Due to these assumption the evolution of the state variable can be defined according to the differential equation

$$\dot{x}(t) = f(x(t), u_1(t), \dots, u_N(t), t), \quad x(0) = x_0,$$

where $x_0 \in \mathbb{R}$ is the initial value of the state. Furthermore the system dynamics $f(\cdot) : \Omega \rightarrow \mathbb{R}^n$ is defined

on the set

$$\Omega = \{(x, u_1, \dots, u_N, t) \mid x \in \mathbb{R}^n, t \in [0, T], u_i \in U_i, i \in \mathcal{N}\},$$

as a jointly piecewise continuous function which is Lipschitz in x , and also possibly Lipschitz in the u_i 's, depending on whether the underlying information structure (IS) is an open loop or closed loop feedback.

We consider two different ISs: Open Loop (OL), where the controls are just functions of the time t and closed-loop state-feedback (FB), where the controls are allowed to be functions of the current value of the state and of time: $u_i(t) = \gamma_i(t; x(t))$. In the latter case, $\gamma_i : [0, T] \times \mathbb{R}^n \rightarrow U_i$ is known as the policy variable of player i . We require each $\gamma_i(\cdot; \cdot)$ to be jointly piecewise continuous in both arguments and additionally Lipschitz in the state variable. We denote the class of all such mappings by Γ_i . We further require that f be Lipschitz not only in x but also in the controls such that the differential equation for the state

$$\dot{x}(t) = f(x(t), \gamma_1(t; x(t)), \dots, \gamma_N(t; x(t)), t), \quad x(0) = x_0,$$

admits a unique piecewise continuously differentiable solution for each $\gamma_i \in \text{Gamma}_i$, $i \in \mathcal{N}$. To capture the OL IS as a special case of this notation, we write $\gamma_i^\eta \in \Gamma_i^\eta$, where η stands for the underlying IS.

All players are cost-minimizers. Let $F_i : \Omega \rightarrow \mathbb{R}$ be player i 's instantaneous (running) cost function, and $S_i : \mathbb{R}^n \rightarrow \mathbb{R}$ the terminal value function. Then the objective function of player i can be given (in accordance with the definitions above) by

$$L_i(u) = \int_0^T F_i(x(t), u_1(t), \dots, u_N(t), t) dt + S_i(x(T)),$$

when $T < \infty$, and

$$L_i(u) = \int_0^\infty F_i(x(t), u_1(t), \dots, u_N(t), t) dt,$$

when $T = \infty$ and $u := \{u_1, \dots, u_N\}$. Substituting $u_i(t) = \gamma_i(t; x(t))$ in the cost function above, we obtain the normal or strategic form of the differential game (DG). Let us denote this new cost function representation by J_i , $i \in \mathcal{N}$, which we write more explicitly as $J_i(\gamma^\eta)$, where $\gamma^\eta := \{\gamma_1^\eta, \dots, \gamma_N^\eta\} \in \text{Gamma}^\eta := \Gamma_1^\eta \times \dots \times \text{Gamma}_N^\eta$.

Let $\gamma_{-i}^\eta := (\gamma_1^\eta, \dots, \gamma_{i-1}^\eta, \gamma_{i+1}^\eta, \dots, \gamma_N^\eta)$, then when we fix γ_{-i}^η to $\gamma_{-i}^{\eta*}$, player i is confronted with the following dynamic optimization (optimal control) problem:

$$\begin{aligned} \min_{\gamma_i \in \Gamma_i^\eta} J_i(\gamma_i, \gamma_{-i}^{\eta*}) &:= \int_0^T F_i(x, \gamma_i(\eta), \gamma_{-i}^{\eta*}(\eta), t) dt + S_i(x(T)) \\ \text{s.t. } \dot{x}(t) &= f(x, \gamma_i(\eta), \gamma_{-i}^{\eta*}(\eta), t), \quad x(0) = x_0. \end{aligned} \tag{OC}_i$$

For $T = \infty$ we just set $S_i \equiv 0$. If we denote the solution to (OC_i) by $\gamma_i^{\text{eta}*}$ and carry out the optimization for each $i \in \mathcal{N}$, then we derive a Nash equilibrium for the given IS η . We summarize this in a more precise way in the following definition.

Definition 1 (η -Nash equilibrium). *For a DG with IS η , the policy N -tuple $\{\gamma_i^{\eta*}, i \in \mathcal{N}\} =: \gamma^{\eta*}$ is an η -Nash equilibrium if, for each $i \in \mathcal{N}$, $\gamma_i^{\eta*}$ solves the optimal control problem (OC_i) . Let $\Gamma^{\eta*}$ be the set of all η -Nash equilibria, as a subset of Γ^η .*

Now let $J_i^{\eta*}$, $i \in \mathcal{N}$, denote the achieved values of the objective functions of the players under a particular η -Nash equilibrium $\gamma^{\eta*}$, and a corresponding total cost achieved (as a convex combination of the individual costs) be given by $J_\mu^{\eta*} = \sum_{i \in \mathcal{N}} \mu_i J_i^{\eta*}$, where μ_i is a positive weighting factor on player i 's cost, satisfying the normalization condition $\sum_{i \in \mathcal{N}} \mu_i = 1$. We assume, without loss of generality, that $J_i^{\eta*} > 0$ for all $i \in \mathcal{N}$, and hence $J_\mu^{\text{eta}*} > 0$.

Next we consider the case of full coordination as a benchmark. Hence we assume that all players agree on minimizing a single objective function which is a convex combination of the individual cost functions. The corresponding underlying optimization problem is the following optimal control problem:

$$\begin{aligned} \min_{\gamma \in \Gamma^\eta} \sum_{i=1}^N \mu_i \left\{ \int_0^T F_i(x, \gamma(\eta), t) dt + S_i(x(T)) \right\} \\ \text{s.t. } \dot{x}(t) = f(x, \gamma(\eta), t), \quad x(0) = x_0. \end{aligned} \quad (\text{COC})$$

where the optimization could also be carried out with respect to u , since the problem is deterministic and not strategic. Thus, the optimal value of (COC) is independent of the IS, which we denote by J_μ° , and the corresponding optimal control by $u^\circ = [u_1^\circ, \dots, u_N^\circ]$. Note that we necessarily have $0 < J_\mu^\circ \leq J_\mu^{\eta^*}$ for any $\gamma^{\eta^*} \in \Gamma^{\eta^*}$. Now we have introduced all the necessary notation to give a precise definition of the PoA.

Definition 2 (Price of Anarchy (PoA)). *Consider an N -person DG as above and its (COC) with $J_\mu^\circ > 0$. The price of anarchy for the DG is*

$$\rho_{N,\mu,T}^\eta = \max_{\gamma^{\eta^*} \in \Gamma^{\eta^*}} \frac{J_\mu^{\eta^*}}{J_\mu^\circ},$$

i.e., the worst-case ratio of the total game cost to the optimum social cost.

Note that the PoA as defined above is lower-bounded by 1. Next we give a precise definition of the PoI.

Definition 3 (Price of Information (PoI)). *Let η_1 and η_2 be two ISs. Consider two N -person DGs which differ only in terms of their ISs, with game 1 having IS η_1 , and game 2 having IS η_2 . Let the values of a particular μ convex combination of the objective functions be $J_\mu^{\eta_1^*}$ and $J_\mu^{\eta_2^*}$, respectively, achieved under the Nash equilibria $\gamma^{\eta_1^*}$ and $\gamma^{\eta_2^*}$. The price of information between the two ISs (under cost minimization) is given by*

$$\chi_{\eta_1}^{\eta_2}(\mu) = \max_{\gamma^{\eta_2^*} \in \Gamma^{\eta_2^*}} J_\mu^{\eta_2^*} / \max_{\gamma^{\eta_1^*} \in \Gamma^{\eta_1^*}} J_\mu^{\eta_1^*}$$

The PoI compares the worst-case costs under two different ISs for the same convex combination, and quantifies the relative loss or gain when the DG is played under a different IS. Finally note that the PoA and the PoI are connected as follows:

$$\chi_{\eta_1}^{\eta_2}(\mu) = \frac{\rho_{N,\mu,T}^{\eta_2}}{\rho_{N,\mu,T}^{\eta_1}}.$$

The analysis of the PoA and the PoI is very difficult for general DGs as there often exists more than one Nash equilibrium, which shows strong dependence on the underlying IS. However the analysis is tractable for specific game structures like scalar linear quadratic DGs with OL or FB IS. This kind of games has many applications in economics and communication networks [1, 9]. In this special class of DGs the players are not interested in the control actions pursued by the other players. For such scalar DGs it is possible to obtain some analytic bounds for the prices of anarchy, information and cooperation. Hence let finally the infinite-horizon scalar N -person linear quadratic DG be defined as

$$L_i(u) = \int_0^\infty [q_i x^2(t) + r_i u_i^2(t)] dt, \quad i \in \mathcal{N}, \quad (2.1)$$

$$\dot{x}(t) = ax(t) + \sum_{i=1}^N b_i u_i(t), \quad x(0) = x_0, \quad (2.2)$$

where $a, b_i \neq 0, q_i > 0, r_i > 0, x_0 \neq 0$ are scalar quantities.

Chapter 3

Some Corrections for the Bounds on the Prices of Anarchy and Information

In this section we want to point out errors made in the proofs of two results (Corollary 1 and Theorem 6) in [8]. While it is quite easy to correct the bounds in Corollary 1, we cannot give any mended results associated to Theorem 6. Thus we will discuss shortly the consequences of the faultiness of Theorem 6 for two further results in [8] (Theorem 8 and Corollary 3) building on it. Finally we will give a list of typos in [8].

Let us start with introducing some variables that are used in Corollary 1 and Theorem 6 (in the order of their first appearance below):

$$\begin{aligned} \sigma_{\max} &= \max_{1 \leq i \leq N} \sigma_i, \quad \Omega \subset \mathcal{N}, \quad n_{\Omega} = |\Omega|, \quad s^{\bullet} = \sum_{i=1}^N \frac{s_i}{\min_{j \in \mathcal{N}} s_j}, \quad \mu_{\max}^s = \max_{i \in \mathcal{N}} \frac{\mu_i}{s_i}, \\ \bar{q} &= \sum_{i=1}^N \mu_i q_i, \quad \mu_{\min}^s = \min_{i=1}^N \frac{\mu_i}{s_i}, \quad \bar{p} = \sum_{i=1}^N p_j, \quad \sigma_i = s_i q_i, \quad s_i = \frac{b_i^2}{r_i}, \quad \bar{\sigma} = \sum_{i=1}^N \sigma_i. \end{aligned}$$

Corollary 1. The first error occurs in the proof of Corollary 1 on page 61, where an upper bound for the numerator of the price of anarchy ρ_{μ}^{FB} is deduced. The second to last equation is given by

$$\begin{aligned} \varrho(M) + a &\leq \max \left\{ \max_{1 \leq n_{\Omega} \leq N} \frac{(2a + \sigma_{\max} - 1)n_{\Omega} + N}{2n_{\Omega} - 1}, 2a + N - 1 \right\} \\ &\leq \max \{2a + N + \sigma_{\max} - 1, 2a + N - 1\} \\ &\leq 2a + N + \sigma_{\max} - 1. \end{aligned} \tag{3.1}$$

As $RS_1 = N + a$ (see also M_2 and M_3 from the application in flow control on page 67 in [8]), the first line of (3.1) has to be corrected to

$$\varrho(M) + a \leq \max \left\{ \max_{1 \leq n_{\Omega} \leq N} \frac{(2a + \sigma_{\max} - 1)n_{\Omega} + N}{2n_{\Omega} - 1}, 2a + N \right\}$$

Next it is argued that the second inequality in (3.1) holds because the quantity

$$\frac{(2a + \sigma_{\max} - 1)n_{\Omega} + N}{2n_{\Omega} - 1} \tag{3.2}$$

increases with n_Ω . But this is only true if

$$2N < 1 - 2a - \sigma_{\max}. \quad (3.3)$$

holds, otherwise (3.2) decreases for increasing $n_\Omega \neq 0$. Condition (3.3) can be obtained from the first derivation of (3.2) with respect to n_Ω that is given by (after some basic simplifications):

$$\frac{1 - 2a - \sigma_{\max} - 2N}{(2n_\Omega - 1)^2}$$

Now for $1 \leq n_\Omega \leq N$ (3.2) can be correctly approximated by

$$\begin{aligned} & \max_{n_\Omega \in \{1, N\}} \frac{(2a + \sigma_{\max} - 1)n_\Omega + N}{2n_\Omega - 1} \\ & \leq \max\{2a + \sigma_{\max} - 1 + N, 2a + \sigma_{\max}\} \leq 2a + \sigma_{\max} - 1 + N. \end{aligned}$$

In summary the numerator of ρ_μ^{FB} can be bounded by

$$\varrho(M) + a \leq \max\{2a + N + \sigma_{\max} - 1, 2a + N\} \leq 2a + N + \sigma_{\max}.$$

This results in corrected upper bounds for the price of anarchy ρ_μ^{FB} for given μ :

$$\begin{aligned} \rho_\mu^{FB} & \leq \left(1 + \frac{1}{2a}(N + \sigma_{\max})\right) s^\bullet, \quad a \neq 0, \\ \rho_\mu^{FB} & \leq \frac{\mu_{\max}^s}{\sqrt{q}\mu_{\min}^s} \sqrt{N}(N + \sigma_{\max}), \quad a = 0. \end{aligned}$$

Theorem 6. There are two errors in the proof of Theorem 6 when approximating the term $\bar{p} - a$. The first equation on page 63 is given by

$$\begin{aligned} \bar{p} - a & = \frac{\bar{p} - a}{N - 1} \left(\sum_{i=1}^N \sqrt{1 - \frac{\sigma_i}{(\bar{p} - a)^2}} + a \right) \\ & = \frac{\bar{p} - a}{N - 1} \left[\frac{N\bar{\sigma}}{2(\bar{p} - a)^2} \left(1 + O\left(\frac{\sigma_{\max}}{2(\bar{p} - a)^2}\right) \right) + a \right]. \end{aligned} \quad (3.4)$$

The first minor error (that has no impact on the overall approximation) appears in the first line of (3.4). The corrected version of the first line reads

$$\bar{p} - a = \frac{\bar{p} - a}{N - 1} \left(\sum_{i=1}^N \sqrt{1 - \frac{\sigma_i}{(\bar{p} - a)^2}} + \frac{a}{\bar{p} - a} \right). \quad (3.5)$$

Applying the Taylor series expansion

$$\begin{aligned} \sqrt{1+x} & = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(1-2n)(n!)^2 (4^n)} x^n \\ & = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \dots, \quad |x| \leq 1. \end{aligned} \quad (3.6)$$

to (3.5) yields

$$\begin{aligned} & \frac{\bar{p} - a}{N - 1} \left(\sum_{i=1}^N \left(1 - \frac{\sigma_i}{2(\bar{p} - a)^2} + O\left(\frac{\sigma_i^2}{(\bar{p} - a)^4}\right) \right) + \frac{a}{\bar{p} - a} \right) \\ & = \frac{\bar{p} - a}{N - 1} \left(N - \frac{\bar{\sigma}}{2(\bar{p} - a)^2} + \sum_{i=1}^N O\left(\frac{\sigma_i^2}{(\bar{p} - a)^4}\right) + \frac{a}{\bar{p} - a} \right). \end{aligned} \quad (3.7)$$

But now there is no way to obtain the second line of (3.4) from (3.7) independent of the minor error pointed out in (3.5).

Note that numerical examples like the application in flow control in [8] can give neither support nor disproof for the results in Theorem 6 as this theorem states asymptotic results.

Consequences. Due to the errors in the proof of Theorem 6, Theorem 8 and Corollary 3 cannot be maintained. Note that Corollary 3 comprehends another error. It implicitly contains the assumption (by using the results of Theorem 8) that N is large but makes a statement for small N : “When $N \geq 3$, the open-loop IS yields better total optimal cost; otherwise the FB information does better.” This statement is even disproved by the price of information $\chi_{FB}^{OL} = 0.9184$ for $N = 2$ from the multiuser rate-based flow control example on pages 66–68.

Further Typos. Finally in Table 3.1 we give a list of further typos in [8].

Position	Original Expression	Corrected Expression	Reference
page 57, below equation (12)	$p := [1, k_1, k_2, \dots, \prod_{i=1}^N k_i]^\top$	$k := [1, k_1, k_2, \dots, \prod_{i=1}^N k_i]^\top$	
page 57, Theorem 2	$\lambda > \sigma_{\max}$	$\lambda^2 > \sigma_{\max}$	[12, Lemma 8.11]
page 59, line 1	$\sigma_i \geq 0$	$\sigma_i > 0$	
page 59, equation (19)	\geq	$>$	
page 60, equation (24)	$\max_k \frac{s_i k_i}{\sum_{i=1}^N s_i k}$	$\max_k \frac{\sum_{i=1}^N s_i k_i}{\sum_{i=1}^N s_i k}$	
page 61, middle	Gersgorin	Gershgorin	
page 62, equation (29)	$\frac{2a}{b\mu_{\max}^s} \geq$	$\frac{2a}{b\mu_{\max}^s} =$	
page 62, Theorem 6		(iii) and (iv) are equal	
page 62, equation (30)	$1 + O\left(\frac{\sigma_i}{(\bar{p}-a)^2}\right)$	$1 + O\left(\frac{\sigma_i}{(\bar{p}-a)}\right)$	
page 63, below equation (33)	$\sigma_{\max} \ll \sigma$	$\sigma_{\max} \ll \bar{\sigma}$	
page 64, Theorem 7	$J_i = k_i^* x_0$	$J_i = k_i^* x_0^2$	[12, Theorem 7.29]
page 64, equation (38)	$k_i = \sigma_i + p_i^2 / (2s_i(\bar{p} - a))$	$k_i = (\sigma_i + p_i^2) / (2s_i(\bar{p} - a))$	
page 64, above equation (40)	35	(35)	
page 65, below equation (45)	(39)	(42)	
page 65, second to last equation	\geq	\leq	$\langle x, y \rangle \leq \ x\ \ y\ $
page 68, second equation	J_{OL}^*	J_{FB}^*	
page 68, second equation	J_{FB}^*	J_{OL}^*	
page 68, middle	$\frac{3}{8}$	$\frac{5}{16}$	
page 70, Table 1, last entry	$\sqrt{2 - \frac{1}{N}} \left(\frac{1}{2} + \frac{1}{N}\right)$	$\sqrt{2 - \frac{1}{N}} \left(\frac{1}{2} + \frac{1}{2N}\right)$	
page 70, equation (55)	$\sqrt{2 - \frac{1}{N}} \left(\frac{1}{2} + \frac{1}{N}\right)$	$\sqrt{2 - \frac{1}{N}} \left(\frac{1}{2} + \frac{1}{2N}\right)$	
page 70, below equation (55)	$f(N) = \frac{1}{N}$	$f(N) = N$	
page 72, Table 2, first column	$\frac{1}{N}$	N	
page 72, below Table 2	$f(N) = \frac{1}{N}$	$f(N) = N$	

Table 3.1: List of typos in [8].

Chapter 4

Bounds for the Price of Cooperation

In this section we deduce bounds for an adapted variant of the price of cooperation for scalar linear quadratic DGs with OL IS. For simplicity of the exposition we first consider the two-player case and later on present generalized results for the N-player case.

At first we recall the definition of the price of cooperation proposed in [8]: Let $\lambda_i := \{\lambda_i^j, j \in \mathcal{N}\}$ be a set of nonnegative parameters adding up to 1 and let $\tilde{J}_i(\gamma^\eta; \lambda_i)$, $i \in \mathcal{N}$, be defined by

$$\tilde{J}_i(\gamma^\eta; \lambda_i) := \sum_{j=1}^N \lambda_i^j J_j(\gamma^\eta), \quad i \in \mathcal{N}.$$

Consider the η IS DG with cost functions \tilde{J} 's, and let $\tilde{\Gamma}^\eta$ be the set of all its η -Nash equilibria. For $\tilde{\gamma}^\eta \in \tilde{\Gamma}^\eta$, player i achieves an actual cost of $J_i(\tilde{\gamma}^\eta)$, which may be lower or higher than $J_i^{\eta*}$ defined in Section 2. Note that if $\lambda_i^j = \mu_i$ for all $i, j \in \mathcal{N}$, then all players have the same cost function, and every η -Nash equilibrium solution of the altruistic game is a solution to COC. Now let us give a precise definition of the PoC.

Definition 4 (Price of Cooperation (PoC)). *Consider an N-player DG with a fixed IS η , and with a fixed set of cooperation vectors $\lambda := \{\lambda_i, i \in \mathcal{N}\}$. Let \tilde{J}_i , $i \in \mathcal{N}$, and $\tilde{\Gamma}^\eta$ be as defined above, and Γ^η be the set of all Nash equilibria of the original game. Then, the price of cooperation for player i under the cooperation scheme λ is given by*

$$\nu_i^\eta(\lambda) = \frac{\max_{\gamma \in \tilde{\Gamma}^\eta} J_i(\gamma)}{\max_{\gamma \in \Gamma^\eta} J_i(\gamma)}.$$

For simplicity we assume that player 1 sticks to his objective function (and does not cooperate)

$$\tilde{J}_1(\gamma^\eta; \lambda_1) = J_1(\gamma^\eta) = L_1(u) = \int_0^\infty [q_1 x^2(t) + r_1 u_1^2(t)] dt, \quad \lambda_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

while player 2 cooperates by placing weight μ on the objective function of player 1, hence $\lambda_2 = \begin{pmatrix} \mu \\ 1 - \mu \end{pmatrix}$, $0 < \mu \leq 1$. The adapted objective function of player 2 is given by

$$\begin{aligned} \tilde{J}_2(\gamma^\eta; \lambda_2) &= \mu J_1(\gamma^\eta) + (1 - \mu) J_2(\gamma^\eta) = \mu L_1(u) + (1 - \mu) L_2(u) \\ &= \int_0^\infty [(\mu q_1 + (1 - \mu) q_2) x^2(t) + \mu r_1 u_1^2(t) + (1 - \mu) r_2 u_2^2(t)] dt. \end{aligned}$$

But now the control of player 1 is contained in the integrand of the objective function of player 2. Hence the cooperating player faces a non-standard optimization problem and we are not able to obtain any explicit bounds on the price of cooperation for either player.

In the following we work with a simpler (but still meaningful) version of the price of cooperation that allows us to avoid the difficulties described above: Let $\hat{J}_i(\gamma^\eta; \lambda_i)$, $i \in \mathcal{N}$, be defined by

$$\hat{J}_i(\gamma^\eta; \lambda_i) := \int_0^\infty \left(\sum_{j=1}^N \lambda_i^j q_j \right) x^2(t) + r_i u_i^2(t) dt, \quad i \in \mathcal{N}.$$

Consider the η IS DG with cost functions \hat{J} 's, and let $\hat{\Gamma}^\eta$ be the set of all its η -Nash equilibria.

Definition 5 (Simple Price of Cooperation (sPoC)). *Consider an N -player scalar DG with a fixed IS η , fixed parameters $p = \{\bar{a}, \bar{b}_i, \bar{q}_i, \bar{r}_i, i \in \mathcal{N}\}$, and with a fixed set of cooperation vectors $\lambda := \{\lambda_i, i \in \mathcal{N}\}$. Let \hat{J}_i , $i \in \mathcal{N}$, and $\hat{\Gamma}^\eta$ be as defined above, and Γ^η be the set of all Nash equilibria of the original game. Then, the simple price of cooperation for player i under the cooperation scheme λ is given by*

$$\tau_i^\eta(\lambda, p) = \frac{\max_{\gamma \in \hat{\Gamma}^\eta} J_i(\gamma)}{\max_{\gamma \in \Gamma^\eta} J_i(\gamma)}.$$

An economic reason for working with the sPoC might e.g. be that the cooperating player does only observe the state variables of the other players but not their control variables. Note that this assumption matches nicely with the fact that in scalar linear quadratic DGs the players are not interested in the control actions pursued by the other players.

In the remaining of the section we assume that the IS in the entire DG is OL. Hence each player knows only the value of the initial state of the system. Since the cost runs from zero to infinity, we are interested in controls that yield finite costs. Accordingly, we restrict the controls of the players to belong to the set

$$\mathcal{U}^{OL}(x_0) = \{u \in \mathcal{L}_2[0, \infty) \mid J_i(x_0, u) < \infty, \forall i \in \mathcal{N}\},$$

where $\mathcal{L}_2[0, \infty)$ is the space of square-integrable functions on $[0, \infty)$. Now the following theorem states that Γ^η and $\hat{\Gamma}^\eta$ are singletons for linear quadratic DGs.

Theorem 1 (Open-Loop NE, [6, 8, 12]). *Consider the N -person linear quadratic DG in (2.1) and (2.2), and assume that there exists a unique solution ξ^* to the set of equations*

$$0 = 2a\xi_i + q_i - \xi_i \left(\sum_{j=1}^N s_j \xi_j \right),$$

such that $a - \sum_{j=1}^N s_j \xi_j^* < 0$, where $s_i := b_i^2/r_i$. Then, the game admits a unique open-loop Nash equilibrium for every initial state, given by

$$u_i^*(t) = -\frac{b_i}{r_i} \xi_i^* \exp \left[\left(a - \sum_{j=1}^N s_j \xi_j^* \right) t \right] x_0.$$

The optimal cost to player i using u_i^* is $J_i^* = k_i^* x_0^2$, where k_i^* is the unique solution to

$$2 \left(a - \sum_{j=1}^N s_j \xi_j^* \right) k_i + q_i + s_i (\xi_i^*)^2 = 0,$$

given by

$$k_i^* = \frac{1}{\sqrt{a^2 + \bar{\sigma}}} \left(\frac{q_i}{2} + \frac{\sigma_i q_i}{2(\sqrt{a^2 + \bar{\sigma}} - a)} \right).$$

If player 2 considers \hat{J}_2 instead of \tilde{J}_2 when cooperating, the optimal costs of player 1 are, according to Theorem 1, $J_1(\gamma) = \hat{k}_1^* x_0^2$, $\gamma \in \hat{\Gamma}^\eta$, with

$$\hat{k}_1^* = \frac{1}{\sqrt{a^2 + \hat{\sigma}}} \left(\frac{q_1}{2} + \frac{\sigma_1 q_1}{2(\sqrt{a^2 + \hat{\sigma}} - a)^2} \right),$$

$$\hat{\sigma} = \frac{b_1^2 q_1}{r_1} + \frac{b_2^2(\mu q_1 + (1 - \mu)q_2)}{r_2}.$$

The costs of player 2 cannot be made explicit in general. By applying Theorem 1 we obtain

$$J_2(\gamma) = \int_0^\infty \left[q_2 + \frac{b_2^2(\hat{\xi}_2^*)^2}{r_2} \right] x^2(t) dt$$

$$= \hat{k}_2^* x_0^2 + \mu(q_2 - q_1) \int_0^\infty x^2(t) dt, \quad \gamma \in \hat{\Gamma}^\eta.$$

For $q_1 \neq q_2$ we further have

$$J_2(\hat{\gamma}) > J_2(\gamma), \quad \hat{\gamma} \in \hat{\Gamma}^\eta, \quad \gamma \in \Gamma^\eta$$

as $\gamma \in \Gamma^\eta$ is (per definition of the Nash-equilibrium) the cost minimal strategy of player 2 with respect to his objective function J_2 (when player 1 simultaneously minimizes J_1). We now capture all this in the corollary below.

Corollary 2. *In a two-player scalar DG with open-loop information structure and*

$$\lambda_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} \mu \\ 1 - \mu \end{pmatrix}, \quad 0 < \mu \leq 1, \quad q_1 \neq q_2,$$

the sPoC is given by

$$\tau_1^{OL}(\lambda, p) = \frac{\hat{k}_1^*}{k_1^*} = \frac{\sqrt{a^2 + \hat{\sigma}} [(\sqrt{a^2 + \hat{\sigma}} - a)^2 (\sqrt{a^2 + \hat{\sigma}} - a)^2 + \sigma_1 (\sqrt{a^2 + \hat{\sigma}} - a)^2]}{\sqrt{a^2 + \hat{\sigma}} [(\sqrt{a^2 + \hat{\sigma}} - a)^2 (\sqrt{a^2 + \hat{\sigma}} - a)^2 + \sigma_1 (\sqrt{a^2 + \hat{\sigma}} - a)^2]},$$

$$\tau_2^{OL}(\lambda, p) = \frac{\hat{k}_2^*}{k_2^*} + \frac{\mu(q_2 - q_1) \int_0^\infty x^2(t) dt}{k_2^* x_0^2} > 1.$$

Let us state two numerical examples. If player 2 accounts for both cost coefficients of the state value in the same way ($\mu = \frac{1}{2}$), then the sPoC of player 1 is given by $\tau_1^{OL}(\lambda, p) = \frac{39 - 24\sqrt{2}}{16\sqrt{2}(3 - \sqrt{8})} \approx 1.303$ for $b_1 = q_1 = r_1 = 1$, $b_2 = q_2 = r_2 = a = 2$ and $\tau_1^{OL}(\lambda, p) = \frac{2\sqrt{3}(161 - 14\sqrt{26} - 46\sqrt{6} + 8\sqrt{39})}{\sqrt{13}(161 - 22\sqrt{26} - 30\sqrt{6} + 8\sqrt{39})} \approx 0.882$ for $b_1 = q_1 = r_1 = 2$, $b_2 = q_2 = r_2 = a = 1$.

In the following theorem we show that for $a = 0$ the sPoC can be expressed in a much simpler way.

Theorem 3. *In a two-player scalar DG with open-loop information structure and*

$$a = 0, \quad \lambda_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} \mu \\ 1 - \mu \end{pmatrix}, \quad 0 < \mu \leq 1,$$

the sPoC of player 1 is given by

$$\tau_1^{OL}(\lambda, p) = \frac{(\delta + \alpha\beta^2)^{1.5} (2\delta + (\alpha + \mu - \alpha\mu)\beta^2)}{(2\delta + \alpha\beta^2)(\delta + (\alpha + \mu - \alpha\mu)\beta^2)^{1.5}}, \quad (4.1)$$

where

$$\beta b_1 = b_2, \quad \beta \neq 0, \quad \delta r_1 = r_2, \quad \delta > 0, \quad \alpha q_1 = q_2, \quad \alpha > 0. \quad (4.2)$$

Proof. Using the equations (4.2), the variables $\bar{\sigma}$ and $\hat{\sigma}$ can easily be expressed as

$$\bar{\sigma} = \left(1 + \frac{\alpha\beta^2}{\delta}\right) \sigma_1, \quad \hat{\sigma} = \left(1 + \frac{(\alpha + \mu - \alpha\mu)\beta^2}{\delta}\right) \sigma_1. \quad (4.3)$$

Applying Corollary 2, using (4.3) and doing some basic transformations gives the stated sPoC:

$$\begin{aligned} \tau_1^{OL}(\lambda, p) &= \frac{\hat{k}_1^*}{k_1^*} = \frac{\sqrt{\bar{\sigma}} [\hat{\sigma}\bar{\sigma} + \sigma_1\bar{\sigma}]}{\sqrt{\hat{\sigma}} [\hat{\sigma}\bar{\sigma} + \sigma_1\hat{\sigma}]} = \\ &= \frac{\frac{1}{\sqrt{\left(1 + \frac{(\alpha + \mu - \alpha\mu)\beta^2}{\delta}\right) \sigma_1}} \left(\frac{q_1}{2} + \frac{\sigma_1 q_1}{2\left(1 + \frac{(\alpha + \mu - \alpha\mu)\beta^2}{\delta}\right) \sigma_1}\right)}{\frac{1}{\sqrt{\left(1 + \frac{\alpha\beta^2}{\delta}\right) \sigma_1}} \left(\frac{q_1}{2} + \frac{\sigma_1 q_1}{2\left(1 + \frac{\alpha\beta^2}{\delta}\right) \sigma_1}\right)} = \\ &= \sqrt{\frac{\delta + \alpha\beta^2}{\delta + (\alpha + \mu - \alpha\mu)\beta^2} \frac{(\delta + \alpha\beta^2)(2\delta + (\alpha + \mu - \alpha\mu)\beta^2)}{(2\delta + \alpha\beta^2)(\delta + (\alpha + \mu - \alpha\mu)\beta^2)}}. \end{aligned} \quad (4.4)$$

□

Hence if player 2 accounts for both cost coefficients of the state value in the same way ($\mu = \frac{1}{2}$), a is set to 0 and α is set to 2, then the sPoC of player 1 is given by $\tau_1^{OL}(\lambda, p) = \sqrt{\frac{5}{4}} \cdot \frac{25}{24} \approx 1.165$ for $\beta = \delta = 2$ and $\tau_1^{OL}(\lambda, p) = \sqrt{\frac{6}{5}} \cdot \frac{21}{20} \approx 1.150$ for $\beta = \delta = 1$. If we set $\alpha = \frac{1}{2}$ instead, then the sPoC of player 1 is given by $\tau_1^{OL}(\lambda, p) = \sqrt{\frac{10}{11}} \cdot \frac{95}{99} \approx 0.915$ for $\beta = \delta = \frac{1}{2}$ and $\tau_1^{OL}(\lambda, p) = \sqrt{\frac{6}{7}} \cdot \frac{33}{35} \approx 0.873$ for $\beta = \delta = 1$.

Next we deduce simple overall bounds on the sPoC for general scalar DGs. To formulate the according theorems in a compact way, we introduce another notion.

Definition 6 (Tight Bounds). *The bounds $a < \tau_1^{OL}(\lambda, p) < b$ on the sPoC of a scalar DG are called tight, if there exist sets of parameters and cooperation vectors (p_1, λ_1) and (p_2, λ_2) such that*

$$\tau_1^{OL}(\lambda_1, p_1) - \varepsilon < a,$$

and

$$\tau_1^{OL}(\lambda_2, p_2) + \varepsilon > b,$$

hold for any given $\varepsilon > 0$.

In the following theorem we show that $\left(\frac{\bar{\sigma}}{\hat{\sigma}}\right)^2$ is a tight bound for the sPoC given in Corollary 2.

Theorem 4. *In a two-player scalar DGs with open-loop information structure and*

$$\lambda_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} \mu \\ 1 - \mu \end{pmatrix}, \quad 0 < \mu \leq 1,$$

the sPoC for player 1 is bounded by

$$\begin{aligned} 1 < \tau_1^{OL}(\lambda, p) < \left(\frac{\bar{\sigma}}{\hat{\sigma}}\right)^2, \quad \alpha > 1, \\ \left(\frac{\bar{\sigma}}{\hat{\sigma}}\right)^2 < \tau_1^{OL}(\lambda, p) < 1, \quad 0 < \alpha < 1, \end{aligned} \quad (4.5)$$

and all bounds are tight.

Proof. With the help of (4.3) we find that $\bar{\sigma} < \hat{\sigma}$ for $\alpha < 1$ and $\bar{\sigma} > \hat{\sigma}$ for $\alpha > 1$. Let us w.l.o.g. assume $\alpha > 1$. (The case $0 < \alpha < 1$ can be handled completely analogically.) Using Corollary 2 and $\bar{\sigma} < \hat{\sigma}$ we obtain

$$1 < \tau_1^{OL}(\lambda, p) = g(a) \cdot h(a) =: f(a), \quad (4.6)$$

with

$$g(a) := \frac{\sqrt{a^2 + \bar{\sigma}}}{\sqrt{a^2 + \hat{\sigma}}}, \quad (4.7)$$

$$h(a) := \frac{(\sqrt{a^2 + \hat{\sigma}} - a)^2 (\sqrt{a^2 + \bar{\sigma}} - a)^2 + \sigma_1 (\sqrt{a^2 + \bar{\sigma}} - a)^2}{(\sqrt{a^2 + \bar{\sigma}} - a)^2 (\sqrt{a^2 + \hat{\sigma}} - a)^2 + \sigma_1 (\sqrt{a^2 + \hat{\sigma}} - a)^2}. \quad (4.8)$$

The limes of $f(a)$ for $a \rightarrow \infty$ can be deduced as the product

$$\lim_{a \rightarrow \infty} g(a) \cdot \lim_{a \rightarrow \infty} h(a).$$

For $g(a)$ we clearly have

$$\lim_{a \rightarrow \infty} g(a) = 1. \quad (4.9)$$

With the help of the Taylor series expansion (3.6) we have

$$\left(\sqrt{1 + \frac{x}{a^2}} - 1 \right)^2 = \frac{\bar{x}}{4a^4} + O\left(\frac{1}{a^6}\right). \quad (4.10)$$

Using (4.10) we can rewrite $h(a)$ as

$$h(a) = \frac{\frac{\bar{\sigma}^2}{4a^6} \sigma_1 + O\left(\frac{1}{a^8}\right)}{\frac{\hat{\sigma}^2}{4a^6} \sigma_1 + O\left(\frac{1}{a^8}\right)}.$$

Hence

$$\lim_{a \rightarrow \infty} h(a) = \left(\frac{\bar{\sigma}}{\hat{\sigma}} \right)^2. \quad (4.11)$$

Alternatively we could apply L'Hôpital's rule to $h(a)$ twice to obtain the above result. Now we can combine (4.9) and (4.11) to

$$\lim_{a \rightarrow \infty} f(a) = \left(\frac{\bar{\sigma}}{\hat{\sigma}} \right)^2. \quad (4.12)$$

Next we examine if $f(a) \stackrel{?}{<} \left(\frac{\bar{\sigma}}{\hat{\sigma}} \right)^2$ holds $\forall a \in \mathbb{R}$ and $0 < \hat{\sigma} < \bar{\sigma}$. Using (4.6)-(4.8) we transform the inequality in question to

$$\begin{aligned} \hat{\sigma}^2 \sqrt{a^2 + \bar{\sigma}} \left[(\sqrt{a^2 + \hat{\sigma}} - a)^2 + \sigma_1 \right] (\sqrt{a^2 + \bar{\sigma}} - a)^2 &\stackrel{?}{<} \\ \bar{\sigma}^2 \sqrt{a^2 + \hat{\sigma}} \left[(\sqrt{a^2 + \bar{\sigma}} - a)^2 + \sigma_1 \right] (\sqrt{a^2 + \hat{\sigma}} - a)^2. & \end{aligned}$$

We further split up the above inequality into two simpler ones:

$$\begin{aligned} \hat{\sigma}^2 \sqrt{a^2 + \bar{\sigma}} \left(\sqrt{a^2 + \hat{\sigma}} - a \right)^2 \left(\sqrt{a^2 + \bar{\sigma}} - a \right)^2 &\stackrel{?}{<} \\ \bar{\sigma}^2 \sqrt{a^2 + \hat{\sigma}} \left(\sqrt{a^2 + \bar{\sigma}} - a \right)^2 \left(\sqrt{a^2 + \hat{\sigma}} - a \right)^2, & \end{aligned} \quad (4.13)$$

and

$$\sigma_1 \hat{\sigma}^2 \sqrt{a^2 + \bar{\sigma}} \left(\sqrt{a^2 + \bar{\sigma}} - a \right)^2 \stackrel{?}{<} \sigma_1 \bar{\sigma}^2 \sqrt{a^2 + \hat{\sigma}} \left(\sqrt{a^2 + \hat{\sigma}} - a \right)^2. \quad (4.14)$$

Let us first examine (4.13) that simplifies to

$$\hat{\sigma}^2 \sqrt{a^2 + \bar{\sigma}} \stackrel{?}{<} \bar{\sigma}^2 \sqrt{a^2 + \hat{\sigma}}. \quad (4.15)$$

The terms on both sides of the inequality are > 0 . Thus we can square it and retain the inequality sign. Expanding, simplifying and dividing by $(\bar{\sigma} - \hat{\sigma})$ yields

$$a^2(\bar{\sigma}^2 - \hat{\sigma}^2)(\bar{\sigma} + \hat{\sigma}) \stackrel{?}{>} -\bar{\sigma}\hat{\sigma}(\bar{\sigma}^2 + \bar{\sigma}\hat{\sigma} + \hat{\sigma}^2), \quad (4.16)$$

which clearly holds as the left hand side of the inequality is > 0 and the right hand side is < 0 . Expanding and simplifying (4.14) yields

$$2a\bar{\sigma}^2(a^2 + \hat{\sigma}) - 2a\hat{\sigma}^2(a^2 + \bar{\sigma}) \stackrel{?}{<} \bar{\sigma}^2(2a^2 + \hat{\sigma})\sqrt{a^2 + \hat{\sigma}} - \hat{\sigma}^2(2a^2 + \bar{\sigma})\sqrt{a^2 + \bar{\sigma}}.$$

The terms on both sides of the inequality are > 0 . Squaring, expanding and simplifying the resulting expressions gives

$$\begin{aligned} 2(2a^2 + \bar{\sigma})(2a^2 + \hat{\sigma})\sqrt{a^2 + \bar{\sigma}}\sqrt{a^2 + \hat{\sigma}} &\stackrel{?}{<} \\ \bar{\sigma}^2(a^2 + \hat{\sigma}) + \hat{\sigma}^2(a^2 + \bar{\sigma}) + 8a^2(a^2 + \bar{\sigma})(a^2 + \hat{\sigma}). \end{aligned}$$

Again the terms on both sides of the inequality are > 0 . Squaring, expanding and simplifying yields

$$2\bar{\sigma}^2\hat{\sigma}^2(a^2 + \bar{\sigma})(a^2 + \hat{\sigma}) \stackrel{?}{<} \bar{\sigma}^4(a^2 + \hat{\sigma})^2 + \hat{\sigma}^4(a^2 + \bar{\sigma})^2,$$

which is satisfied due to

$$[\bar{\sigma}^2(a^2 + \hat{\sigma}) - \hat{\sigma}^2(a^2 + \bar{\sigma})]^2 > 0.$$

As (4.12) holds we further have

$$\forall \varepsilon > 0 \exists \lambda_1, p_1 : \tau_1^{OL}(\lambda_1, p_1) + \varepsilon > \left(\frac{\bar{\sigma}}{\hat{\sigma}} \right)^2,$$

and hence the upper bound is tight. It also holds that

$$\lim_{a \rightarrow -\infty} f(a) = 1,$$

and thus

$$\forall \varepsilon > 0 \exists \lambda_2, p_2 : \tau_1^{OL}(\lambda_2, p_2) - \varepsilon < \left(\frac{\bar{\sigma}}{\hat{\sigma}} \right)^2.$$

Alternatively we have $\bar{\sigma} \rightarrow \hat{\sigma}$ for $\mu \rightarrow 0$ and as a consequence

$$\lim_{\mu \rightarrow 0} \tau_1^{OL}(\lambda, p) \rightarrow 1.$$

□

Note that for $\alpha > 1$ the sPoC for both players is > 1 . We further want to mention an interesting observation we made when trying to prove Theorem 4: If we substitute $\tau_1^{OL}(\lambda, p)$ by $h(a)$ from (4.8) in Theorem 4 the associated result clearly gets weaker. But proving this weaker result is gets more involved

than the proof of Theorem 4. Hence in this case multiplying $h(a)$ by a term > 1 ($g(a)$ from (4.7) to be precise) facilitates the proof of the stated bounds.

Let us again take a look at two numerical examples. Note that the given bounds are valid for any a . If player 2 accounts for both cost coefficients of the state value in the same way ($\mu = \frac{1}{2}$), then the sPoC of player 1 is bounded by $1 < \tau_1^{OL}(\lambda, p) < \left(\frac{5}{4}\right)^2 = 1.5625$ for $b_1 = q_1 = r_1 = 1$, $b_2 = q_2 = r_2 = 2$. For $\mu = \frac{1}{2}$, $b_1 = q_1 = r_1 = 2$, $b_2 = q_2 = r_2 = 1$ we obtain the bounds $0.826 \approx \left(\frac{10}{11}\right)^2 < \tau_1^{OL}(\lambda, p) < 1$.

To allow for further analysis, we rewrite the bounds given in Theorem 4 as follows.

Theorem 5. *In a two-player scalar DG with open-loop information structure and*

$$\lambda_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} \mu \\ 1 - \mu \end{pmatrix}, \quad 0 < \mu \leq 1,$$

the sPoC of player 1 is bounded by

$$\begin{aligned} 1 < \tau_1^{OL}(\lambda, p) < \left(\frac{\delta + \alpha\beta^2}{\delta + (\alpha + \mu - \alpha\mu)\beta^2} \right)^2, \quad \alpha > 1, \\ \left(\frac{\delta + \alpha\beta^2}{\delta + (\alpha + \mu - \alpha\mu)\beta^2} \right)^2 < \tau_1^{OL}(\lambda, p) < 1, \quad 0 < \alpha < 1, \end{aligned} \tag{4.17}$$

where

$$\beta b_1 = b_2, \quad \beta \neq 0, \quad \delta r_1 = r_2, \quad \delta > 0, \quad \alpha q_1 = q_2.$$

All bounds are tight.

Proof. Applying (4.3) to (4.5) and simplifying the resulting expressions gives (4.17). \square

Next we give bounds on the sPoC for general scalar DGs that depend only on μ and α .

Theorem 6. *In a two-player scalar DG with open-loop information structure and*

$$\lambda_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} \mu \\ 1 - \mu \end{pmatrix}, \quad 0 < \mu \leq 1, \quad \alpha q_1 = q_2,$$

the sPoC of player 1 is bounded by

$$\begin{aligned} 1 < \tau_1^{OL}(\lambda, p) < \left(\frac{\alpha}{\alpha + \mu - \alpha\mu} \right)^2, \quad \alpha > 1, \\ \left(\frac{\alpha}{\alpha + \mu - \alpha\mu} \right)^2 < \tau_1^{OL}(\lambda, p) < 1, \quad 0 < \alpha < 1. \end{aligned} \tag{4.18}$$

All bounds are tight.

Proof. Using that

$$\frac{z+x}{z+y} > 1, \quad x > y > 0, \tag{4.19}$$

decreases for increasing $z \neq 0$ yields

$$\begin{aligned} 1 < \left(\frac{\delta + \alpha\beta^2}{\delta + (\alpha + \mu - \alpha\mu)\beta^2} \right)^2 < \left(\frac{\alpha\beta^2}{(\alpha + \mu - \alpha\mu)\beta^2} \right)^2, \quad \alpha > 1, \\ \left(\frac{\alpha\beta^2}{(\alpha + \mu - \alpha\mu)\beta^2} \right)^2 < \left(\frac{\delta + \alpha\beta^2}{\delta + (\alpha + \mu - \alpha\mu)\beta^2} \right)^2 < 1, \quad 0 < \alpha < 1. \end{aligned}$$

Now cancelling by β^2 gives (4.18). \square

Hence if player 2 accounts for both cost coefficients of the state value in the same way ($\mu = \frac{1}{2}$), then the sPoC of player 1 is bounded by $1 < \tau_1^{OL}(\lambda) < (\frac{4}{3})^2 \approx 1.778$ for $\alpha = 2$ and $0.444 \approx (\frac{2}{3})^2 < \tau_1^{OL}(\lambda) < 1$ for $\alpha = \frac{1}{2}$.

In the next two theorems we deduce bounds on the sPoC for general scalar DGs that depend only on α and μ respectively.

Theorem 7. *In a two-player scalar DG with open-loop information structure the sPoC of player 1 is bounded by*

$$\begin{aligned} 1 < \tau_1^{OL}(\lambda, p) < \alpha^2, & \quad \alpha > 1, \\ \alpha^2 < \tau_1^{OL}(\lambda, p) < 1, & \quad 0 < \alpha < 1, \end{aligned}$$

where $\alpha q_1 = q_2$. All bounds are tight.

Proof. The first derivative of

$$\frac{\alpha}{\alpha + \mu(1 - \alpha)} \tag{4.20}$$

with respect to μ is given by

$$\frac{-\alpha(1 - \alpha)}{(\alpha + \mu - \alpha\mu)^2},$$

which is < 0 for $0 < \alpha < 1$ and > 0 for $\alpha > 1$. Hence under our constraint $\mu \in (0, 1]$, (4.20) attains its maximum at $\mu = 1$ for $\alpha > 1$ and its minimum at $\mu = 1$ for $0 < \alpha < 1$. \square

Using the above result we can bound the sPoC of player 1 by $1 < \tau_1^{OL}(\lambda) < 4$ for $\alpha = 2$ and $0.25 < \tau_1^{OL}(\lambda) < 1$ for $\alpha = \frac{1}{2}$.

Theorem 8. *In a two-player scalar DG with open-loop information structure and*

$$\lambda_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} \mu \\ 1 - \mu \end{pmatrix}, \quad 0 < \mu \leq 1, \quad \alpha q_1 = q_2,$$

the sPoC of player 1 is bounded by

$$\begin{aligned} 1 < \tau_1^{OL}(\lambda, p) < \left(\frac{1}{1 - \mu} \right)^2, & \quad \alpha > 1, \\ 0 < \tau_1^{OL}(\lambda, p) < 1, & \quad 0 < \alpha < 1. \end{aligned}$$

All bounds are tight.

Proof. The first derivation of the bounds for the sPoC (stated in (4.18))

$$b(\alpha) := \left(\frac{\alpha}{\alpha + \mu - \alpha\mu} \right)^2,$$

with respect to α is given by

$$\frac{2\alpha\mu}{(\alpha + \mu - \alpha\mu)^3}.$$

As this term is > 0 for $\alpha > 0$ and $\mu > 0$

$$\lim_{\alpha \rightarrow \infty} b(\alpha) = \left(\frac{1}{1 - \mu} \right)^2,$$

gives an upper bound on the sPoC for $\alpha > 1$ and

$$\lim_{\alpha \rightarrow 0} b(\alpha) = 0,$$

gives a lower bound on the sPoC for $0 < \alpha < 1$. \square

Hence if player 2 accounts for both cost coefficients of the state value in the same way ($\mu = \frac{1}{2}$), then the sPoC of player 1 is bounded by $1 < \tau_1^{OL}(\lambda) < 4$ for any $\alpha > 1$.

In the following theorem we deduce an overall scalar lower bound on the sPoC that depends on δ and β .

Theorem 9. *In a two-player scalar DGs with open-loop information structure and*

$$a \leq 0, \quad \lambda_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} \mu \\ 1 - \mu \end{pmatrix}, \quad 0 < \mu \leq 1,$$

the sPoC of player 1 is bounded by

$$\left(\frac{\delta}{\delta + \beta^2} \right)^2 < \tau_1^{OL}(\lambda) < 1, \quad 0 < \alpha < 1, \quad (4.21)$$

where

$$\beta b_1 = b_2, \quad \beta \neq 0, \quad \delta r_1 = r_2, \quad \delta > 0, \quad \alpha q_1 = q_2.$$

The bounds are tight.

Proof. The first derivation of

$$\frac{\delta + \alpha\beta^2}{\delta + (\alpha + \mu - \alpha\mu)\beta^2}, \quad (4.22)$$

with respect to α is given by

$$\frac{\mu(\beta^4 + \delta\beta^2)}{(\delta + (\alpha + \mu - \alpha\mu)\beta^2)^2}. \quad (4.23)$$

(4.23) is > 0 for $\mu > 0$ and $\delta > 0$ and hence the inequality

$$\left(\frac{\delta}{\delta + \mu\beta^2} \right)^2 < \left(\frac{\delta + \alpha\beta^2}{\delta + (\alpha + \mu - \alpha\mu)\beta^2} \right)^2, \quad (4.24)$$

holds for $\alpha > 0$. Now the left hand side of (4.24) decreases with increasing $\mu > 0$, hence we obtain (4.21). \square

Applying the above result for $\delta = \beta = 1$ yields $0.25 = \frac{1}{4} < \tau_1^{OL}(\lambda) < 1$ for any $\alpha \in (0, 1)$.

The following theorem indicates that we can deduce strengthened overall bounds on the sPoC for general scalar DGs with $a \leq 0$.

Theorem 10. *In a two-player scalar DGs with open-loop information structure and*

$$a \leq 0, \quad \lambda_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} \mu \\ 1 - \mu \end{pmatrix}, \quad 0 < \mu \leq 1,$$

the sPoC for player 1 is bounded by

$$\begin{aligned} 1 < \tau_1^{OL}(\lambda, p) &\leq \frac{(\delta + \alpha\beta^2)^{1.5}(2\delta + (\alpha + \mu - \alpha\mu)\beta^2)}{(2\delta + \alpha\beta^2)(\delta + (\alpha + \mu - \alpha\mu)\beta^2)^{1.5}}, \quad \alpha > 1, \\ \frac{(\delta + \alpha\beta^2)^{1.5}(2\delta + (\alpha + \mu - \alpha\mu)\beta^2)}{(2\delta + \alpha\beta^2)(\delta + (\alpha + \mu - \alpha\mu)\beta^2)^{1.5}} &\leq \tau_1^{OL}(\lambda, p) < 1, \quad \alpha \in (0, 1), \end{aligned} \quad (4.25)$$

where

$$\beta b_1 = b_2, \quad \beta \neq 0, \quad \delta r_1 = r_2, \quad \delta > 0, \quad \alpha q_1 = q_2.$$

All bounds are tight.

Proof. Using (4.6) together with (4.4) we obtain

$$\begin{aligned} f(0) &= g(0) \cdot h(0) = \sqrt{\frac{\bar{\sigma}}{\hat{\sigma}}} \cdot \frac{(\hat{\sigma}\bar{\sigma} + \sigma_1\bar{\sigma})}{(\hat{\sigma}\bar{\sigma} + \sigma_1\hat{\sigma})} \\ &= \sqrt{\frac{\delta + \alpha\beta^2}{\delta + (\alpha + \mu - \alpha\mu)\beta^2}} \cdot \frac{(\delta + \alpha\beta^2)(2\delta + (\alpha + \mu - \alpha\mu)\beta^2)}{(2\delta + \alpha\beta^2)(\delta + (\alpha + \mu - \alpha\mu)\beta^2)}. \end{aligned}$$

Let us w.l.o.g. assume $\alpha > 1$. (The case $0 < \alpha < 1$ can be handled completely analogically.) Then we aim to show that

$$f(a) \stackrel{?}{<} f(0), \quad a < 0, \alpha > 1.$$

Using (4.7) together with (4.19) yields

$$g(a) = \sqrt{\frac{a^2 + \bar{\sigma}}{a^2 + \hat{\sigma}}} < \sqrt{\frac{\bar{\sigma}}{\hat{\sigma}}} = g(0), \quad a < 0, \alpha > 1.$$

Next we examine if

$$h(a) \stackrel{?}{<} \frac{\hat{\sigma}\bar{\sigma} + \sigma_1\bar{\sigma}}{\hat{\sigma}\bar{\sigma} + \sigma_1\hat{\sigma}}, \quad a < 0, \alpha > 1,$$

holds. Applying (4.8) and rearranging the terms we obtain

$$\begin{aligned} & \left[(\sqrt{a^2 + \hat{\sigma}} - a)^2 (\sqrt{a^2 + \bar{\sigma}} - a)^2 + \sigma_1 (\sqrt{a^2 + \bar{\sigma}} - a)^2 \right] (\hat{\sigma}\bar{\sigma} + \sigma_1\hat{\sigma}) \\ & \stackrel{?}{<} \left[(\sqrt{a^2 + \bar{\sigma}} - a)^2 (\sqrt{a^2 + \hat{\sigma}} - a)^2 + \sigma_1 (\sqrt{a^2 + \hat{\sigma}} - a)^2 \right] (\hat{\sigma}\bar{\sigma} + \sigma_1\bar{\sigma}). \end{aligned}$$

After expanding and some further simplifications, we can split up the inequality above into

$$\begin{aligned} & (\sqrt{a^2 + \hat{\sigma}} - a)^2 (\sqrt{a^2 + \bar{\sigma}} - a)^2 (\bar{\sigma} - \hat{\sigma}) \\ & \stackrel{?}{>} \hat{\sigma}\bar{\sigma} \left[(\sqrt{a^2 + \bar{\sigma}} - a)^2 - (\sqrt{a^2 + \hat{\sigma}} - a)^2 \right], \end{aligned} \tag{4.26}$$

and

$$\bar{\sigma} (\sqrt{a^2 + \hat{\sigma}} - a)^2 \stackrel{?}{>} \hat{\sigma} (\sqrt{a^2 + \bar{\sigma}} - a)^2. \tag{4.27}$$

Expanding and simplifying (4.26) yields

$$\begin{aligned} & (a^2 - 2a\sqrt{a^2 + \bar{\sigma}})(a^2 - 2a\sqrt{a^2 + \hat{\sigma}})(\bar{\sigma} - \hat{\sigma}) \\ & + (2a^2\bar{\sigma} + 2a^2\hat{\sigma})(\bar{\sigma} - \hat{\sigma}) - 2a(\bar{\sigma}^2\sqrt{a^2 + \hat{\sigma}} - \hat{\sigma}^2\sqrt{a^2 + \bar{\sigma}}) \stackrel{?}{>} 0. \end{aligned}$$

The first two addends are > 0 as all involved multiplicands are > 0 . The third addend is also > 0 , this can be seen by using (4.15) and (4.16). Expanding and simplifying (4.27) gives

$$-a(\bar{\sigma} - \hat{\sigma}) \stackrel{?}{>} \hat{\sigma}\sqrt{a^2 + \bar{\sigma}} - \bar{\sigma}\sqrt{a^2 + \hat{\sigma}}.$$

The left hand side of the inequality is > 0 while the right hand side is < 0 because

$$\bar{\sigma}\sqrt{a^2 + \hat{\sigma}} > \hat{\sigma}\sqrt{a^2 + \bar{\sigma}},$$

holds. To verify this inequality, we square it, rearrange the terms and divide it by $(\bar{\sigma} - \hat{\sigma})$ to get

$$a^2(\bar{\sigma} + \hat{\sigma}) > -\bar{\sigma}\hat{\sigma}.$$

□

Let us again look at two numerical examples. If player 2 accounts for both cost coefficients of the state value in the same way ($\mu = \frac{1}{2}$), then the sPoC of player 1 is bounded by $1 < \tau_1^{OL}(\lambda, p) < \frac{25\sqrt{10}}{24\sqrt{8}} \approx 1.165$ for $\alpha = \beta = \delta = 2$. For $\mu = \alpha = \beta = \delta = \frac{1}{2}$ we obtain the bounds $0.915 \approx \frac{95\sqrt{10}}{99\sqrt{11}} < \tau_1^{OL}(\lambda, p) < 1$.

Next we state bounds on the sPoC for general scalar DGs with $a \leq 0$ that depend only on μ and α .

Theorem 11. *In a two-player scalar DGs with open-loop information structure and*

$$a \leq 0, \quad \lambda_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} \mu \\ 1 - \mu \end{pmatrix}, \quad 0 < \mu \leq 1, \quad \alpha q_1 = q_2,$$

the sPoC for player 1 is bounded by

$$\begin{aligned} 1 < \tau_1^{OL}(\lambda) \leq A, \quad \alpha > 1, \\ A \leq \tau_1^{OL}(\lambda) < 1, \quad 0 < \alpha < 1, \end{aligned} \tag{4.28}$$

with

$$A := \frac{\left(6\alpha - \mu + \alpha\mu + \sqrt{12\alpha(\alpha + \mu - \alpha\mu) + \mu^2(1 - \alpha)^2}\right)^{1.5}}{\left(6\alpha + 7\mu - 7\alpha\mu + \sqrt{12\alpha(\alpha + \mu - \alpha\mu) + \mu^2(1 - \alpha)^2}\right)^{1.5}} \cdot \frac{\left(2\alpha + 3\mu - 3\alpha\mu + \sqrt{12\alpha(\alpha + \mu - \alpha\mu) + \mu^2(1 - \alpha)^2}\right)}{\left(2\alpha - \mu + \alpha\mu + \sqrt{12\alpha(\alpha + \mu - \alpha\mu) + \mu^2(1 - \alpha)^2}\right)}.$$

All bounds are tight.

Proof. The numerator of the first derivation of

$$\frac{(z+x)^{1.5}(2z+y)}{(2z+x)(z+y)^{1.5}} \tag{4.29}$$

with respect to z is given by

$$\begin{aligned} (z+x)^{0.5}(z+y)^{0.5} [(5z+2x+1.5y)(z+y)(2z+x) - (5z+2y+1.5x) \cdot \\ (z+x)(2z+y)] = (z+x)^{0.5}(z+y)^{0.5}(y-x)(4z^2+z(x+y)-0.5xy). \end{aligned}$$

Hence

$$z^* = \frac{-(x+y) \pm \sqrt{x^2 + 10xy + y^2}}{8}, \tag{4.30}$$

describes the extreme values of (4.29). The second derivation of (4.29) with respect to z is (after some simplifications) given by

$$\begin{aligned} \frac{(y-x) [0.5(z+x)^{-0.5}(4z^2+z(x+y)-0.5xy) + (z+x)^{0.5}(8z+x+y)]}{(2z+x)^4(z+y)^5} - \\ \frac{(z+x)^{0.5}(y-x)(4z^2+z(x+y)-0.5xy)(4(2z+x)(z+y)^{2.5} + 2.5(z+y)^{1.5}(2z+x)^2)}{(2z+x)^4(z+y)^5}. \end{aligned}$$

For z^* from (4.30) the above expression reduces to

$$\frac{(y-x)(z+x)^{0.5}(8z+x+y)}{(2z+x)^4(z+y)^5} \tag{4.31}$$

as

$$4(z^*)^2 + z^*(x+y) - 0.5xy = 0,$$

holds. Now (4.31) is > 0 for z^* (with positive sign) from (4.30), if and only if $y - x > 0$. Setting $x := \alpha\beta^2$ and $y := (\alpha + \mu - \alpha\mu)\beta^2$ we have $y > x$ for $0 < \alpha < 1$ and $x > y$ for $\alpha > 1$. Hence for fixed α , β and μ the fraction from (4.25) (as $a \leq 0$, we can work with the bounds for the sPoC (4.25) stated in Theorem 10) attains its minimum for $0 < \alpha < 1$ and its maximum for $\alpha > 1$ at

$$\delta^* = \frac{\beta^2}{8} \left(\alpha\mu - 2\alpha - \mu + \sqrt{12(\alpha + \mu - \alpha\mu) + \mu^2(1 - \alpha^2)} \right). \quad (4.32)$$

Applying (4.32) to (4.25), canceling the resulting expression by β^5 and simplifying the terms yields (4.28). \square

Hence if player 2 accounts for both cost coefficients of the state value in the same way ($\mu = \frac{1}{2}$), then the sPoC of player 1 is bounded by $1 < \tau_1^{OL}(\lambda) < 1.167$ for $\alpha = 2$ and $0.804 < \tau_1^{OL}(\lambda) < 1$ for $\alpha = \frac{1}{2}$.

Next we state additional bounds on the sPoC for general scalar DGs with $a \leq 0$ that depend only on μ and α . These additional bounds are weaker than the ones from Theorem 11 but they are given by easier expressions. Thus we will make use of them for deducing further bounds later on.

Theorem 12. *In a two-player scalar DGs with open-loop information structure and*

$$a \leq 0, \quad \lambda_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} \mu \\ 1 - \mu \end{pmatrix}, \quad 0 < \mu \leq 1, \quad \alpha q_1 = q_2,$$

the sPoC for player 1 is bounded by

$$\begin{aligned} 1 < \tau_1^{OL}(\lambda) &\leq B, & \alpha > 1, \\ B &\leq \tau_1^{OL}(\lambda) < 1, & 0 < \alpha < 1, \end{aligned} \quad (4.33)$$

with

$$B := \sqrt{\frac{\alpha}{\alpha + \mu - \alpha\mu}} \cdot \frac{\sqrt{2}(3\alpha + \mu - \alpha\mu) + 4\sqrt{\alpha(\alpha + \mu - \alpha\mu)}}{\sqrt{2}(3\alpha + 2\mu - 2\alpha\mu) + 4\sqrt{\alpha(\alpha + \mu - \alpha\mu)}},$$

Proof. As $a \leq 0$, we can work with the bounds for the sPoC (4.25) stated in Theorem 10. Using (4.19) yields

$$\begin{aligned} 1 < \sqrt{\frac{\delta + \alpha\beta^2}{\delta + (\alpha + \mu - \alpha\mu)\beta^2}} &< \sqrt{\frac{\alpha\beta^2}{(\alpha + \mu - \alpha\mu)\beta^2}}, & \alpha > 1, \delta > 0, \\ \sqrt{\frac{\alpha\beta^2}{(\alpha + \mu - \alpha\mu)\beta^2}} &< \sqrt{\frac{\delta + \alpha\beta^2}{\delta + (\alpha + \mu - \alpha\mu)\beta^2}} < 1, & 0 < \alpha < 1, \delta > 0. \end{aligned}$$

For fixed α , β and μ the fraction

$$\frac{(\delta + \alpha\beta^2)(2\delta + (\alpha + \mu - \alpha\mu)\beta^2)}{(2\delta + \alpha\beta^2)(\delta + (\alpha + \mu - \alpha\mu)\beta^2)},$$

attains its minimum for $0 < \alpha < 1$ and its maximum for $\alpha > 1$ at

$$\delta^* = \frac{\beta^2}{\sqrt{2}} \sqrt{\alpha(\alpha + \mu - \alpha\mu)},$$

To see this, we examine the numerator of the first derivation of

$$\frac{(z + x)(2z + y)}{(2z + x)(z + y)}, \quad (4.34)$$

with respect to z that is given by

$$(4z + 2x + y)(2z^2 + 2zy + zx + xy) - (4z + 2y + x) \cdot (2z^2 + 2zx + zy + xy) = 2z^2(y - x) - xy(y - x).$$

Hence

$$z^* = \pm \sqrt{\frac{xy}{2}}, \quad (4.35)$$

describes the extreme values of (4.34). The second derivation of (4.34) with respect to z is (after some simplifications) given by

$$\frac{4z(2z + x)(2z + y)(y - x) - (y - x)(2z^2 - xy)(8z + 4y + 2x)}{(2z + x)^3(z + y)^3}. \quad (4.36)$$

The second addend of the numerator of (4.36) is 0 as $2(z^*)^2 = xy$ holds for z^* from (4.35). Hence (4.36) is > 0 for the z^* with positive sign from (4.35), if and only if $y - x > 0$. Now setting $x := \alpha\beta^2$ and $y := (\alpha + \mu - \alpha\mu)\beta^2$, we have $y > x$ for $0 < \alpha < 1$ and $x > y$ for $\alpha > 1$.

In summary we obtain (after applying some standard operations to simplify the terms)

$$\begin{aligned} 1 < \tau_1^{OL}(\lambda) &\leq \sqrt{\frac{\alpha\beta^2}{(\alpha + \mu - \alpha\mu)\beta^2}} \cdot \frac{\beta^4(\sqrt{\alpha(\alpha + \mu - \alpha\mu)} + \sqrt{2}\alpha)}{\beta^4(2\sqrt{\alpha(\alpha + \mu - \alpha\mu)} + \sqrt{2}\alpha)} \\ &\quad \frac{(2\sqrt{\alpha(\alpha + \mu - \alpha\mu)} + \sqrt{2}(\alpha + \mu - \alpha\mu))}{(\sqrt{\alpha(\alpha + \mu - \alpha\mu)} + \sqrt{2}(\alpha + \mu - \alpha\mu))}, \quad \alpha > 1, \\ 1 > \tau_1^{OL}(\lambda) &\geq \sqrt{\frac{\alpha\beta^2}{(\alpha + \mu - \alpha\mu)\beta^2}} \cdot \frac{\beta^4(\sqrt{\alpha(\alpha + \mu - \alpha\mu)} + \sqrt{2}\alpha)}{\beta^4(2\sqrt{\alpha(\alpha + \mu - \alpha\mu)} + \sqrt{2}\alpha)} \\ &\quad \frac{(2\sqrt{\alpha(\alpha + \mu - \alpha\mu)} + \sqrt{2}(\alpha + \mu - \alpha\mu))}{(\sqrt{\alpha(\alpha + \mu - \alpha\mu)} + \sqrt{2}(\alpha + \mu - \alpha\mu))}, \quad 0 < \alpha < 1. \end{aligned}$$

Cancelling the above fractions by $\beta^5\sqrt{\alpha(\alpha + \mu - \alpha\mu)}$ and expanding yields (4.33). \square

Let us again look at two numerical examples. If player 2 accounts for both cost coefficients of the state value in the same way ($\mu = \frac{1}{2}$), then the sPoC of player 1 is bounded by $1 < \tau_1^{OL}(\lambda) \leq \frac{11\sqrt{2}+8\sqrt{3}}{\sqrt{65+12}} \approx 1.213$ for $\alpha = 2$ and $0.762 \approx \frac{7+4\sqrt{3}}{4\sqrt{6+6\sqrt{2}}} \leq \tau_1^{OL}(\lambda) < 1$ for $\alpha = \frac{1}{2}$.

Next we give overall bounds on the sPoC for general scalar DGs with $a \leq 0$ that depend only on α .

Theorem 13. *In a two-player scalar DGs with open-loop information structure and*

$$a \leq 0, \quad \lambda_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} \mu \\ 1 - \mu \end{pmatrix}, \quad 0 < \mu \leq 1, \quad \alpha q_1 = q_2,$$

the sPoC for player 1 is bounded by

$$\begin{aligned} 1 < \tau_1^{OL}(\lambda) &\leq \frac{\sqrt{2\alpha}(2\alpha + 1) + 4\alpha}{\sqrt{2}(\alpha + 2) + 4\sqrt{\alpha}}, \quad \alpha > 1, \\ \frac{\sqrt{2\alpha}(2\alpha + 1) + 4\alpha}{\sqrt{2}(\alpha + 2) + 4\sqrt{\alpha}} &\leq \tau_1^{OL}(\lambda) < 1, \quad 0 < \alpha < 1. \end{aligned} \quad (4.37)$$

Proof. Under the constraint $\mu \in (0, 1]$ the term (4.20) attains its maximum at $\mu = 1$ for $\alpha > 1$ and its minimum at $\mu = 1$ for $0 < \alpha < 1$. (For details see the proof of Theorem 7.) Hence the inequalities

$$\sqrt{\alpha} < \sqrt{\frac{\alpha}{\alpha + \mu - \alpha\mu}}, \quad 0 < \alpha < 1, \quad 0 < \mu \leq 1. \quad (4.38a)$$

and

$$\sqrt{\frac{\alpha}{\alpha + \mu - \alpha\mu}} < \sqrt{\alpha}, \quad \alpha > 1, \quad 0 < \mu \leq 1, \quad (4.38b)$$

hold. Next we take a look at the term

$$f(\alpha, \mu) := \frac{\sqrt{2}(3\alpha + \mu - \alpha\mu) + 4\sqrt{\alpha(\alpha + \mu - \alpha\mu)}}{\sqrt{2}(3\alpha + 2\mu - 2\alpha\mu) + 4\sqrt{\alpha(\alpha + \mu - \alpha\mu)}}. \quad (4.39)$$

The first derivation of (4.39) with respect to μ is (after some simplifications)

$$\frac{2\alpha(\alpha - 1) \left(\sqrt{2}\alpha(2 - \mu) + \sqrt{2}\mu + 3\sqrt{\alpha(\alpha + \mu - \alpha\mu)} \right)}{\sqrt{\alpha(\alpha + \mu - \alpha\mu)} \left(\sqrt{2}\alpha(3 - 2\mu) + 2\sqrt{2}\mu + 4\sqrt{\alpha(\alpha + \mu - \alpha\mu)} \right)^2}.$$

This term is > 0 for $\alpha > 1$, $0 < \mu \leq 1$ and < 0 for $0 < \alpha < 1$, $0 < \mu \leq 1$. Thus for fixed $\alpha > 1$ the function $f(\alpha, \mu)$ is strictly monotonically increasing for increasing $\mu \in (0, 1]$. For fixed $\alpha \in (0, 1)$ the function $f(\alpha, \mu)$ is strictly monotonically decreasing for increasing $\mu \in (0, 1]$. Hence we obtain the following bounds

$$\begin{aligned} f(\alpha, \mu) &\leq \frac{\sqrt{2}(2\alpha + 1) + 4\sqrt{\alpha}}{\sqrt{2}(\alpha + 2) + 4\sqrt{\alpha}}, \quad \alpha > 1, \\ f(\alpha, \mu) &\geq \frac{\sqrt{2}(2\alpha + 1) + 4\sqrt{\alpha}}{\sqrt{2}(\alpha + 2) + 4\sqrt{\alpha}}, \quad 0 < \alpha < 1. \end{aligned} \quad (4.40)$$

Applying the results from (4.38) and (4.40) to the bounds from Theorem 12 (that we can use due to $a \leq 0$) finally gives (4.37). \square

Applying the above result for $\alpha = 2$ we find that the sPoC of player 1 is bounded by $1 < \tau_1^{OL}(\lambda) \leq \sqrt{2} \frac{9}{8} \approx 1.591$ for any $\mu \in (0, 1)$. Setting $\alpha = \frac{1}{2}$ we have $0.629 \approx \frac{4\sqrt{2}}{9} \leq \tau_1^{OL}(\lambda) < 1$.

In the following theorem we deduce overall lower bounds on the sPoC for general scalar DGs with $a \leq 0$ that depend only on μ .

Theorem 14. *In a two-player scalar DGs with open-loop information structure and*

$$a \leq 0, \quad \lambda_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} \mu \\ 1 - \mu \end{pmatrix}, \quad 0 < \mu \leq 1, \quad \alpha q_1 = q_2,$$

the sPoC for player 1 is bounded by

$$1 < \tau_1^{OL}(\lambda) < \sqrt{\frac{1}{1 - \mu}} \cdot \frac{\sqrt{2}(3 - \mu) + 4\sqrt{1 - \mu}}{\sqrt{2}(3 - 2\mu) + 4\sqrt{1 - \mu}}, \quad \alpha > 1. \quad (4.41)$$

Proof. The first derivation of (4.20) with respect to α is given by

$$\frac{\mu}{(\alpha + \mu - \alpha\mu)^2},$$

which is > 0 for $\mu > 0$. Hence the inequality

$$\sqrt{\frac{\alpha}{\alpha + \mu - \alpha\mu}} < \sqrt{\frac{1}{1 - \mu}}, \quad 0 < \mu \leq 1, \quad (4.42)$$

holds. The first derivation of (4.39) with respect to α is (after some simplifications)

$$\frac{2\mu \left(\sqrt{2}\alpha(2 - \mu) + \sqrt{2}\mu + 3\sqrt{\alpha(\alpha + \mu - \alpha\mu)} \right)}{\sqrt{\alpha(\alpha + \mu - \alpha\mu)} \left(\sqrt{2}\alpha(3 - 2\mu) + 2\sqrt{2}\mu + 4\sqrt{\alpha(\alpha + \mu - \alpha\mu)} \right)^2}.$$

This term is > 0 for $\alpha > 0$ and $\mu \in (0, 1]$. Thus it increases for increasing $\alpha > 0$ and we obtain

$$f(\alpha, \mu) < \frac{\sqrt{2}(3 - \mu) + 4\sqrt{1 - \mu}}{\sqrt{2}(3 - 2\mu) + 4\sqrt{1 - \mu}}, \quad \alpha > 0, \quad 0 < \mu \leq 1. \quad (4.43)$$

Applying the results from (4.42) and (4.43) to the bounds from Theorem 12 (that we can use due to $a \leq 0$) finally gives (4.41). \square

For $\mu = \frac{1}{2}$ the sPoC of player 1 is bounded by $1 < \tau_1^{OL}(\lambda) < \sqrt{2\frac{9}{8}} \approx 1.591$ for any $a \leq 0$ and any $\alpha > 1$.

In the following theorem we deduce an overall scalar lower bound for the sPoC for scalar DGs with $a \leq 0$ that depends on δ and β .

Theorem 15. *In a two-player scalar DGs with open-loop information structure and*

$$a \leq 0, \quad \lambda_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} \mu \\ 1 - \mu \end{pmatrix}, \quad 0 < \mu \leq 1,$$

the sPoC of player 1 is bounded by

$$\sqrt{\frac{\delta}{\delta + \beta^2} \frac{2\delta + \beta^2}{2(\delta + \beta^2)}} < \tau_1^{OL}(\lambda) < 1, \quad 0 < \alpha < 1, \quad (4.44)$$

where

$$\beta b_1 = b_2, \quad \beta \neq 0, \quad \delta r_1 = r_2, \quad \delta > 0, \quad \alpha q_1 = q_2.$$

Proof. As $a \leq 0$ we can work with the bounds for the sPoC stated in Theorem 10. The first derivation of (4.22) with respect to α is > 0 for $\mu > 0$ and $\delta > 0$ (for details see the proof of Theorem 9). Hence the inequality

$$\sqrt{\frac{\delta}{\delta + \mu\beta^2}} < \sqrt{\frac{\delta + \alpha\beta^2}{\delta + (\alpha + \mu - \alpha\mu)\beta^2}}, \quad (4.45)$$

holds for $\mu > 0$ and $\delta > 0$. Next observe that the first derivation of

$$t(\alpha) = \frac{(\delta + \alpha\beta^2)(2\delta + (\alpha + \mu - \alpha\mu)\beta^2)}{(2\delta + \alpha\beta^2)(\delta + (\alpha + \mu - \alpha\mu)\beta^2)},$$

with respect to α yields

$$\frac{\mu(1 - \alpha)\beta^4[2\alpha(1 - \mu) + \mu] + \mu\alpha\beta^4[\alpha(1 - \mu) + \mu] + 3\mu\beta^2\delta + 2\mu\delta^2}{(2\delta + \alpha\beta^2)^2(\delta + (\alpha + \mu - \alpha\mu)\beta^2)^2} > 0, \\ 0 < \alpha < 1, \quad \mu \in (0, 1], \quad \delta > 0.$$

Hence the inequality

$$\frac{2\delta + \mu\beta^2}{2(\delta + \mu\beta^2)} < t(\alpha), \quad 0 < \alpha < 1, \quad (4.46)$$

holds for $\mu \in (0, 1]$ and $\delta > 0$. For $\delta > 0$ the left hand sides of (4.45) and (4.46) decrease for increasing μ . Setting $\mu = 1$ and combining (4.45) and (4.46) finally yields (4.44). \square

Applying the above bounds for $\delta = \beta = 1$ gives $0.530 \approx \frac{3}{4\sqrt{2}} < \tau_1^{OL}(\lambda) < 1$ for any $\alpha \in (0, 1)$.

We can further generalize part of the results for two-player scalar linear quadratic DGs to N -player scalar linear quadratic DGs. Let us first state the extensions of Corollary 2 and Theorem 4.

Theorem 16. *In an N -player scalar DGs with open-loop information structure and*

$$\lambda_1 = e_1, \quad \lambda_i = \sum_{j=1}^N \mu_{i,j} e_j, \quad \sum_{j=1}^N \mu_{i,j} = 1, \quad i \in \{2, \dots, N\},$$

the sPoC of player 1 is given by

$$\tau_1^{OL}(\lambda) = \frac{\hat{k}_1^*}{k_1^*} = \frac{\sqrt{a^2 + \bar{\sigma}} [(\sqrt{a^2 + \bar{\sigma}} - a)^2 ((\sqrt{a^2 + \bar{\sigma}} - a)^2 + \sigma_1)]}{\sqrt{a^2 + \hat{\sigma}} [(\sqrt{a^2 + \hat{\sigma}} - a)^2 ((\sqrt{a^2 + \hat{\sigma}} - a)^2 + \sigma_1)]}, \quad (4.47)$$

and can be bounded by

$$\begin{aligned} 1 < \tau_1^{OL}(\lambda, p) < \left(\frac{\bar{\sigma}}{\hat{\sigma}}\right)^2, & \quad \sum_{i=2}^N \frac{b_i^2 \sum_{j=1}^N \mu_{i,j} q_j}{r_i} < \sum_{i=2}^N \frac{b_i^2 q_i}{r_i}, \\ \left(\frac{\bar{\sigma}}{\hat{\sigma}}\right)^2 < \tau_1^{OL}(\lambda, p) < 1, & \quad \sum_{i=2}^N \frac{b_i^2 \sum_{j=1}^N \mu_{i,j} q_j}{r_i} > \sum_{i=2}^N \frac{b_i^2 q_i}{r_i}, \end{aligned} \quad (4.48)$$

where

$$\hat{\sigma} = \frac{b_1^2 q_1}{r_1} + \sum_{i=2}^N \frac{b_i^2 \sum_{j=1}^N \mu_{i,j} q_j}{r_i}.$$

and e_i denotes the vector with a 1 in the i^{th} coordinate and 0's elsewhere. All bounds are tight.

Proof. The proof can be done analogically to the proofs of Corollary 2 and Theorem 4. Note that the conditions on α are replaced by the more complex conditions in (4.48) that now ensure $\bar{\sigma} < \hat{\sigma}$ and $\bar{\sigma} > \hat{\sigma}$ respectively. \square

Let us again look at two numerical examples. In order to evaluate the effect of the number of players on the sPoC, we reuse the numbers from the examples related to Corollary 2 and Theorem 4 and double the number of cooperating players.

Hence we obtain the bounds $1 < \tau_1^{OL}(\lambda, p) < \left(\frac{9}{7}\right)^2 \approx 1.653$ for $b_1 = q_1 = r_1 = 1$, $b_2 = b_3 = q_2 = q_3 = r_2 = r_3 = 2$, $\mu_{2,1} = \mu_{2,2} = \mu_{3,1} = \mu_{3,2} = \frac{1}{2}$. Further setting $a = 2$ yields $\tau_1^{OL}(\lambda, p) \approx 1.235$.

Using the data $b_1 = q_1 = r_1 = 2$, $b_2 = b_3 = q_2 = q_3 = r_2 = r_3 = 1$, $\mu_{2,1} = \mu_{2,2} = \mu_{3,1} = \mu_{3,3} = \frac{1}{2}$, we have $0.735 \approx \left(\frac{6}{7}\right)^2 < \tau_1^{OL}(\lambda, p) < 1$. Further setting $a = 2$ yields $\tau_1^{OL}(\lambda, p) \approx 0.864$.

Next we generalize the results of Theorem 3 and Theorem 10.

Theorem 17. *In an N -player scalar DGs with open-loop information structure and*

$$a \leq 0, \quad \lambda_1 = e_1, \quad \lambda_i = \sum_{j=1}^N \mu_{i,j} e_j, \quad \sum_{j=1}^N \mu_{i,j} = 1, \quad i \in \{2, \dots, N\},$$

the sPoC of player 1 is bounded by

$$\begin{aligned} 1 < \tau_1^{OL}(\lambda, p) &\leq C, & \frac{\sum_{i=2}^N \left(\prod_{\substack{j=2 \\ j \neq i}}^N \delta_j \right) \alpha_i \beta_i^2}{\sum_{i=2}^N \left(\prod_{\substack{j=2 \\ j \neq i}}^N \delta_j \right) \left(\mu_{i,1} + \sum_{j=2}^n \mu_{i,j} \alpha_j \right) \beta_i^2} > 1, \\ C &\leq \tau_1^{OL}(\lambda, p) < 1, & \frac{\sum_{i=2}^N \left(\prod_{\substack{j=2 \\ j \neq i}}^N \delta_j \right) \alpha_i \beta_i^2}{\sum_{i=2}^N \left(\prod_{\substack{j=2 \\ j \neq i}}^N \delta_j \right) \left(\mu_{i,1} + \sum_{j=2}^n \mu_{i,j} \alpha_j \right) \beta_i^2} < 1, \end{aligned} \quad (4.49)$$

where

$$C := \frac{\left[\prod_{i=2}^N \delta_i + \sum_{i=2}^N \left(\prod_{\substack{j=2 \\ j \neq i}}^N \delta_j \right) \alpha_i \beta_i^2 \right]^{1.5}}{\left[2 \prod_{i=2}^N \delta_i + \sum_{i=2}^N \left(\prod_{\substack{j=2 \\ j \neq i}}^N \delta_j \right) \alpha_i \beta_i^2 \right]} \cdot \frac{\left[2 \prod_{i=2}^N \delta_i + \sum_{i=2}^N \left(\prod_{\substack{j=2 \\ j \neq i}}^N \delta_j \right) (\mu_{i,1} + \sum_{j=2}^n \mu_{i,j} \alpha_j) \beta_i^2 \right]}{\left[\prod_{i=2}^N \delta_i + \sum_{i=2}^N \left(\prod_{\substack{j=2 \\ j \neq i}}^N \delta_j \right) (\mu_{i,1} + \sum_{j=2}^n \mu_{i,j} \alpha_j) \beta_i^2 \right]^{1.5}},$$

$$\beta_i b_1 = b_i, \quad \beta_i \neq 0, \quad \delta_i r_1 = r_i, \quad \delta_i > 0, \quad \alpha_i q_1 = q_i, \quad \alpha > 0, \quad i \in \{2, \dots, N\}.$$

and e_i denotes the vector with a 1 in the i^{th} coordinate and 0's elsewhere. The bounds are tight for $a = 0$.

Proof. For the given data $\bar{\sigma}$ and $\hat{\sigma}$ are easily obtained as

$$\bar{\sigma} = \left(1 + \sum_{i=2}^N \frac{\alpha_i \beta_i^2}{\delta_i} \right) \sigma_1, \quad \hat{\sigma} = \left(1 + \sum_{i=2}^N \frac{(\mu_{i,1} + \sum_{j=2}^n \mu_{i,j} \alpha_j) \beta_i^2}{\delta_i} \right) \sigma_1, \quad (4.50)$$

Applying (4.50) to (4.47) with a set to zero and doing some basic transformations (similar to the ones in (4.4)) yields $\tau_1^{OL}(\lambda, p) = C$ for $a = 0$. Then continuing analogously to the proof of Theorem 10 yields (4.49). \square

Hence we obtain the bounds $1 < \tau_1^{OL}(\lambda, p) < \frac{36\sqrt{9}}{35\sqrt{7}} \approx 1.166$ for $\alpha_2 = \alpha_3 = \beta_2 = \beta_3 = \delta_2 = \delta_3 = 2$, $\mu_{2,1} = \mu_{2,2} = \mu_{3,1} = \mu_{3,2} = \frac{1}{2}$. Using the data $\mu_{2,1} = \mu_{2,2} = \mu_{3,1} = \mu_{3,3} = \alpha_2 = \alpha_3 = \beta_2 = \beta_3 = \delta_2 = \delta_3 = \frac{1}{2}$ we have $0.873 \approx \frac{66\sqrt{3}}{35\sqrt{7}} < \tau_1^{OL}(\lambda, p) < 1$.

Finally we elaborate the counterpart of Theorem 5 for N players.

Theorem 18. *In an N -player scalar DGs with open-loop information structure and*

$$\lambda_1 = e_1, \quad \lambda_i = \sum_{j=1}^N \mu_{i,j} e_j, \quad \sum_{j=1}^N \mu_{i,j} = 1, \quad i \in \{2, \dots, N\},$$

the sPoC of player 1 is bounded by

$$1 < \tau_1^{OL}(\lambda, p) < D^2, \quad \frac{\sum_{i=2}^N \left(\prod_{\substack{j=2 \\ j \neq i}}^N \delta_j \right) \alpha_i \beta_i^2}{\sum_{i=2}^N \left(\prod_{\substack{j=2 \\ j \neq i}}^N \delta_j \right) (\mu_{i,1} + \sum_{j=2}^n \mu_{i,j} \alpha_j) \beta_i^2} > 1, \quad (4.51)$$

$$D^2 < \tau_1^{OL}(\lambda, p) < 1, \quad \frac{\sum_{i=2}^N \left(\prod_{\substack{j=2 \\ j \neq i}}^N \delta_j \right) \alpha_i \beta_i^2}{\sum_{i=2}^N \left(\prod_{\substack{j=2 \\ j \neq i}}^N \delta_j \right) (\mu_{i,1} + \sum_{j=2}^n \mu_{i,j} \alpha_j) \beta_i^2} < 1,$$

where

$$D := \frac{\prod_{i=2}^N \delta_i + \sum_{i=2}^N \left(\prod_{\substack{j=2 \\ j \neq i}}^N \delta_j \right) \alpha_i \beta_i^2}{\prod_{i=2}^N \delta_i + \sum_{i=2}^N \left(\prod_{\substack{j=2 \\ j \neq i}}^N \delta_j \right) (\mu_{i,1} + \sum_{j=2}^n \mu_{i,j} \alpha_j) \beta_i^2},$$

$$\beta_i b_1 = b_i, \quad \beta_i \neq 0, \quad \delta_i r_1 = r_i, \quad \delta_i > 0, \quad \alpha_i q_1 = q_i, \quad \alpha > 0, \quad i \in \{2, \dots, N\}.$$

All bounds are tight.

Proof. Applying (4.50) to the bounds from Theorem 16 and simplifying the resulting expressions gives (4.51). \square

Hence we obtain the bounds $1 < \tau_1^{OL}(\lambda, p) < \left(\frac{9}{7}\right)^2 \approx 1.653$ for $\alpha_2 = \alpha_3 = \beta_2 = \beta_3 = \delta_2 = \delta_3 = 2$, $\mu_{2,1} = \mu_{2,2} = \mu_{3,1} = \mu_{3,2} = \frac{1}{2}$. Using the data $\mu_{2,1} = \mu_{2,2} = \mu_{3,1} = \mu_{3,3} = \alpha_2 = \alpha_3 = \beta_2 = \beta_3 = \delta_2 = \delta_3 = \frac{1}{2}$ we have $0.735 \approx \left(\frac{6}{7}\right)^2 < \tau_1^{OL}(\lambda, p) < 1$.

Chapter 5

Conclusion

In this thesis we argued that the price of cooperation proposed in [8] is hardly tractable even for very simple differential games. As an alternative we introduced the simple price of cooperation that measures the benefit or loss of a player due to altruistic behavior when he cannot observe the actions taken by the other players. This means that the cooperating players only take into account the costs of the other players that result from their state variables (and ignore the costs resulting from the control variables of the other players). This assumption matches nicely with the fact that in scalar linear quadratic differential games the players are not interested in the control actions pursued by the other players. We deduced several (tight) bounds for the simple price of cooperation for a variety of scalar linear quadratic differential games with 2 and N players, respectively.

Additionally we discussed errors occurring in some proofs in [8] and pointed out the consequences of these errors for some of the main results of the corresponding paper. We argued that the asymptotic results associated to prices of anarchy and information in [8] cannot be maintained.

As cooperation, information and altruistic behavior are key concepts for the law, we think that our thesis provides a valuable contribution to the theory of law. In summary we improved the methods to measure the effects of cooperation and information in a dynamic setting.

Future promising work could be to extend the results for the prices of anarchy, information and cooperation for scalar linear-quadratic differential games and to other classes of games like non-scalar ones. Furthermore transferring other measures like the price of leadership to a dynamic setting could be a fruitful area of research. Finally applying the results obtained to specific models (e.g. from communication networks and economics) would be another worthwhile direction of research.

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